Model of a Clean Superconductor Containing a Random Array of Vortices

ROBERT M. CLEARY*

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305
and
Department of Physics and Materials Research Laboratory, University of Illinois, Urbana, Illinois 61801

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Equations for the Green's function and correlation functions of a superconductor containing a random array of vortices are derived. These differ from the equations for a superconductor containing paramagnetic impurities inasmuch as here, the scattering potentials are macroscopic and cannot be treated in Born approximation. The scattering self-energy is approximated by the T matrix of scattering theory, which has been calculated elsewhere for real energies. In particular, the ultrasonic attenuation is derived for phonons of long wavelength ($ql \ll 1$) and low frequencies, at low vortex densities ($B/H_{c2} \ll 1$) near the critical temperature of the superconductor ($T \simeq T_c$). We find that $a_L^{\parallel}(T_c)$, the vortex cross section for ultrasonic attenuation of longitudinal waves propagating parallel to the vortex axis, equals 210 Å, which agrees well with the experimental result of 240 Å in vanadium.

I. INTRODUCTION

WHEN vortices penetrate a superconductor, there is reason to believe that they form a fairly random array provided the intervortex spacing d exceeds the penetration depth $\lambda(T)$, $d > \lambda(T)$. Vortices interact when the superfluid current of one overlaps that of another and calculations indicate that the fluxoids will make up a stable triangular lattice when $d \lesssim \lambda(T)$.[1–4]

In this paper, a formulation is presented of the theory of a BCS superconductor containing a completely random array of vortices.[5] A calculational technique treating a superconductor with atomic impurities is borrowed from the literature.[6] This technique involves summing an infinite set of Feynman diagrams to obtain an integral equation for the electron Green's function and for correlation functions. Our equations differ from those of a superconductor containing atomic impurities inasmuch as the electron-vortex collision cannot be treated in Born approximation. A nonperturbative approach is called for and this necessitates summing more diagrams in the Feynman series.

The theory takes account of scattering of electronic excitations by stationary vortices, and there is experimental evidence for such scattering in clean superconductors at low vortex densities.[7] A calculation of the vortex cross section is complicated by the Bohm-Aharonov (BA) effect which is nontrivial for a line of flux $c\pi/e(ch/2e$ when $\hbar \neq 1)$.[8] In particular, we have shown elsewhere that the total scattering cross section diverges because of the BA effect.[9] Fortunately, the cross section σ_s for particlelike excitations and holelike excitations to change channel is finite and it is this cross section which appears in transport theory.

In Sec. II, a review of the theory of a superconductor containing a single vortex is presented. An equation for the Green's function in momentum space is derived and the equation for a superconductor containing a random array of n_v vortices per unit area is written for comparison. The Dyson equation and leading diagrams in perturbation theory are illustrated in Fig. 1 for comparison with the analytic expression. Section III is devoted to relating the scattering self-energy to the T matrix of scattering theory. This latter function is easier to calculate and is a good approximation for the self-energy, since the superconducting interaction involves electrons in a narrow shell about the Fermi surface.

In Sec. IV, the thermal conductivity of the model is calculated starting with a Kubo formula. Electron-phonon collisions are ignored and electronic relaxation is produced by electron-vortex collisions and electron

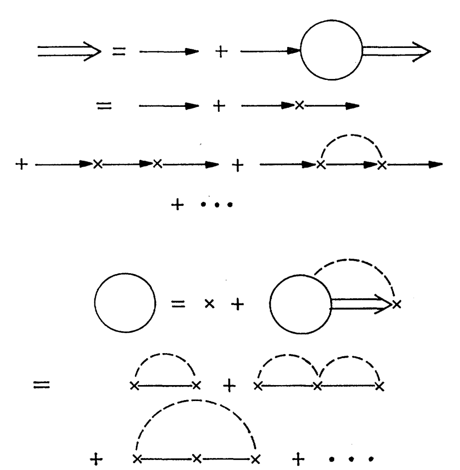

FIG. 1. Diagrammatic equations for Green's function.

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scattering with a residue of spherical impurities. Heat flux is not parallel to the thermal gradient here, even though flux flow is not considered in the model. Electron-vortex scattering has a small left-right asymmetry which produces a small Hall angle, the magnitude of which is estimated to be of order $[p_F\xi(T)]^{-1}$. Section IV investigates the ultrasonic attenuation of the model. In particular, the ultrasonic attenuation is calculated for a low density of vortices, $n_v \to 0$, near the critical temperature of the superconductor. As the vortex density tends to zero, one need only evaluate the T matrix for real energies, and we have calculated this function elsewhere for a superconductor with $\kappa \simeq 1/\sqrt{2}$, where κ is the Ginzburg-Landau parameter. Near T_c, the scattering states are populated sufficiently that one may ignore the bound states of a vortex without incurring too serious an error. The vortex cross section for attenuation of longitudinal sound waves propagating along the vortex axis is estimated to be 210 Å which compares favorably with the experimental determination of 240 Å published by Sinclair and Liebowitz for experiments performed on clean vanadium.[10]

II. GREEN'S FUNCTION

In the theory of superconductivity, electrons are conveniently described by the two-component field operators of Nambu[11]

$$\Psi(\mathbf{x}) = \begin{pmatrix} \psi_\uparrow(\mathbf{x}) \\ \psi_\downarrow^\dagger(\mathbf{x}) \end{pmatrix}, \tag{1}$$

where $\psi_\alpha(\mathbf{x})$ is the annihilation operator for an electron of spin projection α at the point $\mathbf{x}$. For example, the single-particle Green's function is defined by the expression

$$\tilde{\mathcal{G}}(x,x') = -\langle T_\tau \tilde{\Psi}(x) \tilde{\Psi}'(x') \rangle \tag{2}$$

with the thermodynamic Heisenberg operators

$$\tilde{\Psi}(x) = e^{(H-\mu N)\tau} \Psi(\mathbf{x}) e^{-(H-\mu N)\tau}$$

and

$$\tilde{\Psi}'(x) = e^{(H-\mu N)\tau} \Psi^\dagger(\mathbf{x}) e^{-(H-\mu N)\tau}. \tag{3}$$

The angular brackets imply averaging over a grand canonical ensemble and T_τ is the time ordering operator. The Hamiltonian and number operators are H and N, respectively; energies are measured from the Fermi level μ and we take Planck's constant $\hbar$ equal to 1. The Green's function is a 2×2 matrix.

For a single vortex centered at the origin of the (x,y) plane and lying along the z axis, the Green's function satisfies the Gor'kov equation[12]

$$-\left\{\frac{\partial}{\partial\tau} + \tau^{(3)}(2m)^{-1}\left[\left(-i\nabla - \frac{e}{c}\mathbf{A}(r)\tau^{(3)}\right)^2 - p_F^2\right] + \Delta(r)\exp(in\phi\tau^{(3)})\tau^{(1)}\right\}\tilde{\mathcal{G}}(x,x') = \delta^4(x-x'), \tag{4}$$

where

$$\mathbf{x} = (\mathbf{r}, z)$$

and the Pauli spin matrices $\tau^{(i)}$ are introduced to simplify notation. The number of quantized flux units trapped by the vortex is n and the real order parameter $\Delta(r)$ is a function of space, varying most rapidly at its core $r \lesssim \xi(T)$.

The potentials $\mathbf{A}(r)$ and $\Delta(r)$ must be determined self-consistently in the BCS theory. Ampère's law provides a relation between the magnetic field and electric current

$$\nabla \times \mathbf{H}(\mathbf{r}) = 4\pi c^{-1}\langle \mathbf{J}(r)\rangle, \tag{5}$$

where

$$\langle \mathbf{J}(r)\rangle = \lim_{\tau' \to \tau+0} 2[e(2im)^{-1}(\nabla - \nabla')\tilde{\mathcal{G}}_{11}(x,x') - e^2(mc)^{-1}\mathbf{A}(r)\tilde{\mathcal{G}}_{11}(x,x')]_{\mathbf{x}=\mathbf{x}'}.$$

Use has been made here of the fact that the Hamiltonian is not spin-dependent. The self-consistent equation for the gap function is

$$\Delta(r)e^{in\phi} = -\lim_{\tau \to \tau'+0} g\tilde{\mathcal{G}}_{12}(x,x')\big|_{\mathbf{x}=\mathbf{x}'}, \tag{6}$$

where $\tilde{\mathcal{G}}_{12}(x,x')$ is the off-diagonal component of the Green's function, and g is the positive coupling constant of the interaction.

Equation (4) is not rotationally invariant about the z axis. A gauge transformation

$$\mathcal{G}(x,x') = \exp(\tfrac{1}{2}in\phi\tau^{(3)})\tilde{\mathcal{G}}(x,x')\exp(-\tfrac{1}{2}in\phi'\tau^{(3)}) \tag{7}$$

eliminates the phase dependence of the order parameter, so that Eq. (4) becomes

$$-\left\{\frac{\partial}{\partial\tau} + \tau^{(3)}(2m)^{-1}[(-i\nabla + m\mathbf{v}_s(r)\tau^{(3)})^2 - p_F^2] + \Delta(r)\tau^{(1)}\right\}\mathcal{G}(x,x') = \delta^4(x-x'), \tag{8}$$

and the superfluid velocity function is given by

$$m\mathbf{v}_s(r) = n(2r)^{-1} - (e/c)\mathbf{A}(r).$$

This function has a range approximately equal to that of the penetration depth.

For n odd (which includes the important case $n=1$), the new Green's function must be double valued in the (x,y) plane in order that $\tilde{\mathcal{G}}(x,x')$ may be invariant under 2π rotations about the z axis. At present, the discussion shall be limited to the cases where n is even so as to simplify the mathematics. It has been shown how the unusual boundary condition that must be imposed on $\mathcal{G}(x,x')$ for n odd may be incorporated into the results.[9]

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[10] A. C. E. Sinclair and J. Liebowitz, Phys. Rev. **175**, 596 (1968).
[11] Y. Nambu, Phys. Rev. **117**, 648 (1960).
[12] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 735 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].

The operator acting on the Green's function on the left-hand side of Eq. (8) is divided into a homogeneous part and a potential

$$[D(x)-V(\mathbf{r})]\mathcal{G}(x,x')=\delta^4(x-x'), \quad (9)$$

where

$$D(x)=-\left(\frac{\partial}{\partial\tau}-\frac{\tau^{(3)}}{2m}(\nabla^2+p_F{}^2)+\Delta\tau^{(1)}\right)$$

and

$$V(r)=[\Delta(r)-\Delta]\tau^{(1)}+\tfrac{1}{2}mv_s(r)^2\tau^{(3)} + (2im)^{-1}[\nabla\cdot m\mathbf{v}_s(r)+m\mathbf{v}_s(\mathbf{r})\cdot\nabla]. \quad (10)$$

For large r, $\Delta(r)$ tends to its value in the Meissner state $\Delta(T)$.

The solution of the simpler equation

$$D(x)\mathcal{G}_0(x)=\delta^4(x)$$

is readily obtained by introducing the Fourier transform

$$\mathcal{G}_0(x)=(2\pi)^{-3}T\sum_l\int d\mathbf{p}\exp[i(\mathbf{p}\cdot\mathbf{x}-\omega_l\tau)]\mathcal{G}_0(p,\omega_l), \quad (11)$$

where $\omega_l=(2l+1)\pi T$ and l runs over all positive and negative integers. A simple calculation will show

$$\mathcal{G}_0(p,\omega_l)=-(i\omega_l+\xi_p\tau^{(3)}+\Delta\tau^{(1)})(\omega_l{}^2+\xi_p{}^2+\Delta^2)^{-1}, \quad (12)$$

where

$$\xi_p=(2m)^{-1}(p^2-p_F{}^2)\simeq v_F(p-p_F).$$

Equation (9) may be recast into an integral equation in momentum space. Since the potential depends only on the radial coordinate $\mathbf{r}$, it is necessary to separate the momentum vector into a component confined to the (x,y) plane and another lying along the z axis

$$\mathbf{p}=(\mathbf{q},k).$$

Rewriting Eq. (9) as an integral equation in coordinate space and introducing the Fourier transforms

$$\mathcal{G}(x,x')=(2\pi)^{-3}T\sum_l\int d\mathbf{p}\,d\mathbf{p}' \times e^{i[\mathbf{p}\cdot\mathbf{x}-\mathbf{p}'\cdot\mathbf{x}'-\omega_l(\tau-\tau')]}\mathcal{G}(\mathbf{q},\mathbf{q}';k,\omega_l)\delta(k-k'), \quad (13)$$

and

$$V(\mathbf{q},\mathbf{q}')=\int d\mathbf{r}\,e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})\,e^{i\mathbf{q}'\cdot\mathbf{r}},$$

one may show that

$$\mathcal{G}(\mathbf{q},\mathbf{q}';k,\omega_l)=\mathcal{G}_0(p,\omega_l)\left[\delta(\mathbf{q}-\mathbf{q}') +(2\pi)^{-2}\int d\mathbf{q}_1\,V(\mathbf{q},\mathbf{q}_1)\mathcal{G}(\mathbf{q}_1,\mathbf{q}';k,\omega_l)\right]. \quad (14)$$

It is convenient to introduce a scattering self-energy in connection with the Green's function by defining

$$\mathcal{G}(\mathbf{q},\mathbf{q}';k,\omega_l)=\mathcal{G}_0(p,\omega_l)\delta(\mathbf{q}-\mathbf{q}') +\mathcal{G}_0(p,\omega_l)\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l)\mathcal{G}_0(p',\omega_l), \quad (15)$$

where the self-energy is a solution of

$$\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l)=V(\mathbf{q},\mathbf{q}')+(2\pi)^{-2}\int d\mathbf{q}_1\,V(\mathbf{q},\mathbf{q}_1)\mathcal{G}_0(p_1,\omega_l) \times\mathcal{T}(\mathbf{q}_1,\mathbf{q}';k,\omega_l). \quad (16)$$

Equation (8) may be modified to encompass the case of a superconductor containing a finite density ($n_v=eB/c\pi$) of vortices by replacing $m\mathbf{v}_s(r)$ with $\sum_i m\mathbf{v}_s(|\mathbf{r}-\mathbf{r}_i|)$ and $\Delta(r)$ by $\Delta+\sum_i[\Delta(|\mathbf{r}-\mathbf{r}_i|)-\Delta]$. Here, B is the flux density penetrating the superconductor and $\mathbf{r}_i$ is the radius vector to the axis of the ith vortex in the superconductor. The potentials of a single isolated vortex are employed for each element of the array. Although the $\mathbf{r}_i$ are random, the vortices remain aligned with the z axis, and deviations from perfect alignment due to crystal imperfections, sample size, or any other cause shall not be considered here.

The exact solution of the equations for an infinite number of randomly located vortices is very difficult to obtain. For many purposes, however, such a solution contains much superfluous information. For example, the thermodynamics of the system is insensitive to the exact location of the individual vortices provided an average density is maintained throughout the sample. It is not unreasonable, therefore, to average the position of the vortices in the equations, so that

$$\mathcal{G}(\mathbf{q},\mathbf{q}';k,\omega_l)=\delta(\mathbf{q}-\mathbf{q}')\mathcal{G}(q,k,\omega_l), \quad (17)$$

thereby producing an anisotropic but homogeneous superconductor, and to consider only those properties of the model which are characterized by a long wavelength.

After averaging over the positions $\mathbf{r}_i$ and in the low-density limit, $p_Fl\gg1$, where l is the mean free path of the excitations due to scattering by vortices, the equations are

$$\mathcal{G}(q,k,\omega_l)=\mathcal{G}_0(p,\omega_l)[1+n_v\mathcal{T}(\mathbf{q},\mathbf{q};k,\omega_l)\mathcal{G}(q,k,\omega_l)], \quad (18)$$

and we employ the same symbol as before for the self-energy which now satisfies

$$\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l)=V(\mathbf{q},\mathbf{q}')+(2\pi)^{-2}\int d\mathbf{q}_1\,V(\mathbf{q},\mathbf{q}_1) \times\mathcal{G}(q_1,k,\omega_l)\mathcal{T}(\mathbf{q}_1,\mathbf{q}';k,\omega_l). \quad (19)$$

The corresponding set of diagrams are illustrated in Fig. 1, where the unperturbed Green's function is a single solid line with an arrowhead, the complete Green's function is a double line with an arrowhead, the scattering self-energy is a circle, the potential is a cross, and a factor n_v is associated with every group of crosses joined by a dashed line, including a single isolated cross. Diagrams with intersecting dashed lines are ignored since they are of higher order in $(p_F l)^{-1}$.

Inverting Eq. (18), one has

$$\mathcal{G}^{-1}(q,k,\omega_l)=i\omega_l-\xi_p\tau^{(3)}-\Delta\tau^{(1)}-n_v\mathcal{T}(\mathbf{q},\mathbf{q};k,\omega_l). \quad (20)$$

To determine the Green's function, one makes the ansatz

$$\mathcal{G}^{-1}(q,k,\omega_l)=i\tilde{\omega}_l-\xi_p\tau^{(3)}-\tilde{\Delta}_l\tau^{(1)}, \quad (21)$$

and the renormalized frequency and order parameter are easily identified as

$$\tilde{\omega}_l(q,k)=\omega_l+in_v\mathcal{T}^{(0)}(\mathbf{q},\mathbf{q};k,\omega_l)$$

and

$$\tilde{\Delta}_l(q,k)=\Delta+n_v\mathcal{T}^{(1)}(\mathbf{q},\mathbf{q};k,\omega_l). \quad (22)$$

The notation

$$\mathcal{T}=\sum_{i=0}^{3}\tau^{(i)}\mathcal{T}^{(i)}$$

has been introduced, where $\tau^{(0)}$ is the unit matrix in two dimensions. The component $\mathcal{T}^{(3)}$ is absorbed by the chemical potential, compared to which it is negligible.

The concept of renormalized frequency and order parameter was introduced by Abrikosov and Gor'kov in their original paper on superconducting alloys.[6] The ratio $\tilde{\omega}_l/\tilde{\Delta}_l$ is equal to ω_l/Δ when the scattering potentials are proportional to the matrix $\tau^{(3)}$. This is not so for a superconductor containing paramagnetic impurities or for a thin current-carrying film of alloy material.[13,14] In these cases, as in that of scattering by vortices, there are pair-breaking mechanisms which, for a finite density of scatterers, modify the superconducting state. In the present case, for example, the gap parameter Δ is a functional of $V(\mathbf{r})$ and will differ from its BCS value. It appears as a constant in the free propagator $\mathcal{G}_0(p,\omega_l)$, but the magnitude of Δ is undetermined up to this point. It is necessary, therefore, to reevaluate Δ from the self-consistency condition Eq. (6) which becomes

$$\Delta=gT\sum_l\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{\tilde{\Delta}_l(q,k)e^{-i\omega_l\epsilon}}{\tilde{\omega}_l^2(q,k)+\xi_p^2+\tilde{\Delta}_l^2(q,k)}. \quad (23)$$

The momentum integration is cut off to prevent a divergent result. This is equivalent to limiting the superconducting interaction to electrons in the energy interval

$$-\omega_D<\xi<\omega_D,$$

where ω_D is the Debye frequency of the phonon spectrum.

III. T MATRIX

The T matrix is obtained from the wave functions of the homogeneous part of Eq. (4) known as the Bogoliubov equation.[15] The equation is easier to solve than a singular integral equation for the self-energy. It is desirable to know, therefore, if the self-energy can be approximated by the T matrix. We begin by relating these two quantities exactly.

For a single vortex, the energy eigenfunctions of the Bogoliubov equation satisfy the integral equation

$$\Phi_{\mathbf{q},k}(\mathbf{r})=u_{\mathbf{q},k}(\mathbf{r})+\int d\mathbf{r}'g(\mathbf{r}-\mathbf{r}';k,E+i\epsilon)\times V(\mathbf{r}')\Phi_{\mathbf{q},k}(\mathbf{r}'), \quad (24)$$

where

$$g(\mathbf{r}-\mathbf{r}';k,z)=\mathcal{G}_0(\mathbf{r}-\mathbf{r}';k,\omega_l)\,|_{z=i\omega_l}.$$

The eigenfunctions for the homogeneous superconductor are

$$u_{\mathbf{q},k}(\mathbf{r})=(2\pi)^{-1}\eta e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (25)$$

where

$$q^2=p_F^2-k^2+2m(E^2-\Delta^2)^{1/2} \quad (26)$$

and

$$\bar{u}_{q,k}(\mathbf{r})=(2\pi)^{-1}\tau^{(1)}\eta e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad (27)$$

with

$$\bar{q}^2=p_F^2-k^2-2m(E^2-\Delta^2)^{1/2}. \quad (28)$$

The normalization vector η has components proportional to the BCS coherence factors.[9] The Green's function must be evaluated for $z=E+i\epsilon$, where the positive increment ϵ ensures the correct boundary conditions for the eigenfunctions at infinity.

The T matrix is defined by the integral

$$\tilde{T}_{11}(\mathbf{q},\mathbf{q}',k)=\int d\mathbf{r}\,u_{\mathbf{q},k}^{\dagger}(\mathbf{r})V(\mathbf{r})\Phi_{\mathbf{q}',k}(\mathbf{r}), \quad (29)$$

where q and q' are constrained by Eq. (26). Transitions from channel q to channel $\bar{q}$ are treated in a slightly different way

$$\tilde{T}_{12}(\mathbf{q},-\mathbf{q}',k)=\int d\mathbf{r}\,u_{\mathbf{q},k}^{\dagger}(\mathbf{r})V(\mathbf{r})\Phi_{\mathbf{q}',k}(\mathbf{r}), \quad (30)$$

where q' is constrained by Eq. (28), and a minus sign is introduced before $\bar{q},\bar{q}'$ in the defining equations for later convenience.

Introducing the vector function $\zeta(\mathbf{q},\mathbf{q}',k)$, defined by

$$\tilde{T}(\mathbf{q},\mathbf{q}',k)=(2\pi)^{-2}\eta^{\dagger}\zeta(\mathbf{q},\mathbf{q}',k), \quad (31)$$

one may easily show that it satisfies the integral equation

$$\zeta(\mathbf{q},\mathbf{q}',k)=V(\mathbf{q},\mathbf{q}')\eta'+(2\pi)^{-2}\int d\mathbf{q}_1\,V(\mathbf{q},\mathbf{q}_1)\times g(q_1,k,E+i\epsilon)\zeta(\mathbf{q}_1,\mathbf{q}',k), \quad (32)$$

[13] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **39**, 1781 (1960) [English transl.: Soviet Phys.—JETP **12**, 1243 (1961)].

[14] K. Maki, Progr. Theoret. Phys. (Kyoto) **29**, 10 (1963).

[15] N. N. Bogoliubov, V. V. Tolmachev, and D. V. Shirkov, in *A New Method in the Theory of Superconductivity* (Consultants Bureau Inc., New York, 1959).

and it is apparent that

$$\tilde{T}_{ij}(\mathbf{q}_i,\mathbf{q}_j,k)=(2\pi)^{-2}\eta_i^{\dagger}T(\mathbf{q}_i,\,\mathbf{q}_j;\,k,\,E_j+i\epsilon)\eta_j\,, \tag{33}$$

where

$$T(\mathbf{q}_i,\mathbf{q}_j;\,k,z)=\mathcal{T}(\mathbf{q}_i,\mathbf{q}_j;\,k,\omega_l)\,|_{z=i\omega_l}\,. \tag{34}$$

The energy E_j appearing in Eq. (33) is equal to that of a single-particle excitation of momenta q_j, k.

The magnitude of the momentum $p=(q^2+k^2)^{1/2}$ appearing in the self-energy, Eqs. (21)–(23) of Sec. II, is approximated by its value on the Fermi surface p_F. The self-energy can then be related to the T matrix and is only a function of energy ω and momentum angle

$$\theta=\tan^{-1}q/k\simeq\tan^{-1}\bar{q}/k\,. \tag{35}$$

For example, consider the lifetimes of the excitations, which may be extracted from $\operatorname{Im}(\bar{E}-\bar{\Delta}^2)^{1/2}$, where $\bar{E}$ is the analytic continuation of $\tilde{\omega}$ for the continuous complex variable E. The temperature functions are recovered for $E=i\omega_l$. In the low-density limit, $n_v\to 0$,

$$\begin{aligned}\operatorname{Im}(\bar{E}^2-\bar{\Delta}^2)^{1/2}\sim &-n_v(E^2-\Delta^2)^{-1/2}\\ &\times\operatorname{Im}[ET^{(0)}(\mathbf{q},\mathbf{q};\,k,E)+\Delta T^{(1)}(\mathbf{q},\mathbf{q};\,k,E)]\\ &=-n_vE(E^2-\Delta^2)^{-1/2}\operatorname{Im}[\eta^{\dagger}T(\mathbf{q},\mathbf{q};\,k,E)\eta]\,.\end{aligned} \tag{36}$$

Approximating the matrix elements of T by the T matrix, we have

$$\begin{aligned}\operatorname{Im}(\bar{E}^2-\bar{\Delta}^2)^{1/2}\sim &-n_vE(2\pi)^2(E^2-\Delta^2)^{-1/2}\\ &\times\operatorname{Im}\tilde{T}_{11}(\mathbf{q},\mathbf{q};\,k)\,.\end{aligned} \tag{37}$$

With this expression, it is a simple matter to show that in the limit $n_v\to 0$ [16]

$$\operatorname{Im}(\bar{E}^2-\bar{\Delta}^2)^{1/2}=[2\tau_n(\theta,E)]^{-1}. \tag{38}$$

In Eq. (38) the variables q and k have been replaced by the more convenient ones θ and E. The relaxation time $\tau_n(\theta,E)$ is related to the cross section by

$$\tau_n(\theta,E)^{-1}=n_vqm^{-1}\sigma_n(\theta,E)\,, \tag{39}$$

and $\sigma_n(\theta,E)$ is the total integrated cross section for scattering excitations of momenta q and k by a vortex. In a previous paper, we have shown that $\sigma_n(\theta,E)$ is a divergent quantity because of the BA effect.[9] In calculating transport properties, one must include scattering-in terms to ensure a relaxation time that is finite and independent of the BA effect.

Deviations of the thermodynamics and density of states from their BCS value are gauged by studying the ratio of the frequency E to the order parameter Δ

$$\begin{aligned}E/\Delta=\bar{E}/\bar{\Delta}+&(n_v/\Delta\bar{\Delta})\\ &\times[\bar{\Delta}T^{(0)}(\mathbf{q},\mathbf{q};\,k,E)+\bar{E}T^{(1)}(\mathbf{q},\mathbf{q};\,k,E)]\,.\end{aligned} \tag{40}$$

Making the T-matrix approximation once more, we have

$$E/\Delta=\bar{E}/\bar{\Delta}+(n_v\bar{E}/\Delta\bar{\Delta})(2\pi)^2\tilde{T}_{12}(\mathbf{q},\mathbf{q},k)\,, \tag{41}$$

[16] A. L. Fetter, Phys. Rev. **140**, 1921 (1965).

where the T-matrix element $\tilde{T}_{12}$ is independent of the BA effect, and one need not worry about cutting off divergent integrals.

To estimate how important renormalization effects may be, one may examine the dimensionless parameter $\alpha=(\Delta\tau_s)^{-1}$ which appears in the theory of a superconductor containing paramagnetic impurities.[13] Renormalization effects are important when $\alpha\sim 1$ and taking $\sigma_s\sim\xi(T)$, the temperature-dependent coherence length, one finds that for

$$n_v\sim\xi(T)^{-2}\,, \tag{42}$$

or $H\sim H_{c2}$ renormalization effects are important. This result is of the right order of magnitude, since superconductors have a gapless behavior near the upper critical field.[17] Near H_{c1} the density of states will have a low-lying branch with $E<\Delta$ due to the bound states of a vortex.[18] As H increases beyond H_{c1} a knowledge of the T matrix for complex energies would enable a calculation to be made of the corrections to the density of states $n(E)$ and the order parameter $\Delta(T)$ as a function of B.

IV. THERMAL CONDUCTIVITY

The thermal gradient is a probe of low frequency and long wavelength, which has a linear response in the limit of small amplitude. The heat flow has two components: one due to electrons and another to phonons.[19] Our calculation is limited to the electron transport which is assumed to be limited by collisions with impurities and vortices.

The treatment of Ambegaokar and Griffin lends itself well in the present calculation.[20] The response of the system is provided by the correlation function of the energy current.

In the temperature representation

$$Q(1,2)=\langle T_\tau\mathbf{h}(1)\mathbf{h}(2)\rangle\,, \tag{43}$$

where

$$\begin{aligned}\mathbf{h}(x)=-i(2m)^{-1}(\nabla'\partial/\partial\tau+\nabla\partial/\partial\tau')&\\ \times\textstyle\sum_\alpha\bar{\psi}_\alpha'(x')\bar{\psi}_\alpha(x)\,|_{x=x'}&\,.\end{aligned} \tag{44}$$

In terms of the two-component field operators

$$\mathbf{h}(x)=-i(2m)^{-1}\left(\frac{\nabla'\partial}{\partial\tau}+\frac{\nabla\partial}{\partial\tau'}\right)\bar{\Psi}'(x')\tau^{(3)}\Psi(x)\,|_{x=x'}$$

apart from an irrelevent constant. The gradient operator appearing in Eq. (44) is not a gauge-invariant quantity. One might consider replacing it by the sum $\nabla+i\sum_j mv_s(\mathbf{r}-\mathbf{r}_j)$, which is gauge-invariant. The sum over j, however, is much smaller than ∇ outside of a

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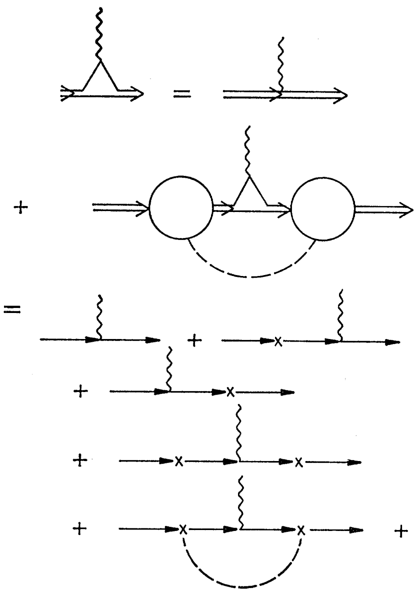

FIG. 2. Diagrammatic equations for the correlation function.

narrow cylinder of approximate radius p_F^{-1} about each vortex axis and it is acceptable, therefore, to ignore it. The matter current accompanying the energy current is of relative order $2mT/p_F^2$, as Ambegaokar and Griffin have shown, and is quite negligible so that Eq. (44) may be employed for the heat flow.

Expressing the correlation function in terms of field operators, one has

$$Q(1,2)=-(4m^2)^{-1}\left(\frac{\nabla_1'\partial}{\partial\tau_1}+\frac{\nabla_1\partial}{\partial\tau_1'}\right)\left(\frac{\nabla_2'\partial}{\partial\tau_2}+\frac{\nabla_2\partial}{\partial\tau_2'}\right)\times\langle T_\tau\bar{\Psi}'(1')\tau^{(3)}\tilde{\Psi}(1)\bar{\Psi}'(2')\tau^{(3)}\tilde{\Psi}(2)\rangle|_{1=1',2=2'}.$$

Some terms proportional to $\delta(t_1-t_2)$ have not been kept. These arise from commuting derivatives of τ_1 and τ_2 with the ordering operator T_τ. It is customary to neglect them as they do not contribute to the thermal conductivity. Later on we shall experience an apparently divergent frequency sum, but the source of the difficulty will be evident.

With a given fixed array of vortices, Wick's theorem simplifies the thermodynamic average over the four fermion field operators appearing in the correlation function

$$Q(1,2)=(4m^2)^{-1}\left(\frac{\nabla_1'\partial}{\partial\tau_1}+\frac{\nabla_1\partial}{\partial\tau_1'}\right)\left(\frac{\nabla_2'\partial}{\partial\tau_2}+\frac{\nabla_2\partial}{\partial\tau_2'}\right)\times\mathrm{Tr}[\tau^{(3)}G(1,2')\tau^{(3)}G(2,1')]_{1=1',2=2'},$$

and Tr takes the trace of a matrix. This equation is Fourier transformed in space and time and in the long-wavelength limit one has

$$Q(\omega_0)=(4m^2)^{-1}T\sum_l\int\frac{d\mathbf{p}\,d\mathbf{p}'}{(2\pi)^3}(\omega_l+\omega_l')^2\mathbf{p}\mathbf{p}'\times\mathrm{Tr}[\tau^{(3)}\mathcal{G}(\mathbf{p},\mathbf{p}',\omega_l')\tau^{(3)}\mathcal{G}(\mathbf{p}'\mathbf{p},\omega_l)],\quad(45)$$

where $\omega_l'=\omega_l+\omega_0$ and $\omega_0=2\nu\pi T$ with ν an integer. The correlation function Q has bosonlike frequencies, since the energy current is bilinear in the electron field operators.

As in the calculation of the Green's function, it is convenient to homogenize the superconductor by taking an average over the position of the vortices in the (x,y) plane. However, the average of the product of two Green's functions is not equal to the product of two averaged Green's functions. One must include scattering-in terms, which constitute a kind of vertex correction. The diagrammatic technique is applicable to the calculation of the product of two Green's functions and the basic diagrams are summarized in Fig. 2. The bare vertex, which is represented by a wavy line appears only once in each diagram, since we are interested in the linear response of the system and limit ourselves to first order in perturbation theory. The scattering-in term is the second on the right of the Dyson's equation in Fig. 2.

The equations corresponding to the diagrams must be solved for a vector function $\mathbf{K}(q,k,\omega_l,\omega_l')$

$$\mathbf{K}(q,k,\omega_l,\omega_l')=\mathcal{G}(q,k,\omega_l')\Big[\mathbf{p}\tau^{(3)}(\omega_l+\omega_l')+n_v(2\pi)^{-2}\int d\mathbf{q}'\,\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l')\times\mathbf{K}(q',k,\omega_l,\omega_l')\mathcal{T}(\mathbf{q}',\mathbf{q};k,\omega_l)\Big]\mathcal{G}(q,k,\omega_l),\quad(46)$$

and the correlation function is obtained by carrying out the frequency sum and momentum integration in the expression below:

$$Q(\omega_0)=(4m^2)^{-1}T\sum_l(2\pi)^{-3}\int d\mathbf{p}\times\mathrm{Tr}[\mathbf{p}(\omega_l+\omega_l')\tau^{(3)}\mathbf{K}(q,k,\omega_l,\omega_l')].\quad(47)$$

Diagrams with intersecting dashed lines are not included in Fig. 2 as they are of higher order in $(p_Fl)^{-1}$. The resulting equation is sometimes referred to as the ladder approximation for $\mathbf{K}(q,k,\omega_l,\omega_l')$. The Green's function which appears in Eq. (46) is that of Sec. II for a superconductor containing a random array of vortices.

The electron legs are removed from the vertex function by introducing two new functions $M(q,k,\omega_l,\omega_l')$ and $\mathbf{N}(q,k,\omega_l,\omega_l')$ in the defining equation

$$\mathbf{K}(q,k,\omega_l,\omega_l')=\mathcal{G}(q,k,\omega_l')\times[M(q,k,\omega_l,\omega_l')\hat{\mathbf{z}}+\mathbf{N}(q,k,\omega_l,\omega_l')]\mathcal{G}(q,k,\omega_l).\quad(48)$$

The vector $\mathbf{N}$ is confined to the (x,y) plane and satisfies the equation

$$\mathbf{N}(q,k,\omega_l,\omega_l') = \tau^{(3)}(\omega_l+\omega_l')\mathbf{q} + n_v(2\pi)^{-2}\int d\mathbf{q}'\, \mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l') \times \mathcal{G}(q',k,\omega_l')\mathbf{N}(q',k,\omega_l,\omega_l') \times \mathcal{G}(q',k,\omega_l)\mathcal{T}(\mathbf{q}',\mathbf{q};k,\omega_l). \quad (49)$$

The function M satisfies a similar equation with k replacing $\mathbf{q}$ in the first term on the right-hand side.

In the Hartree-Fock approximation, the scattering-in term in Eq. (49) is neglected and the correlation function reduces simply to

$$Q(\omega_0) = (4m^2)^{-1}T\sum_l (2\pi)^{-3}\int d\mathbf{p}\,\mathbf{pp}(\omega_l+\omega_l')^2 \times \mathrm{Tr}[\tau^{(3)}\mathcal{G}(q,k,\omega_l')\tau^{(3)}\mathcal{G}(q,k,\omega_l)]. \quad (50)$$

It is evident at this point that the sum over frequency will diverge for large l because of the term $(\omega_l+\omega_l')^2$ which appears in the summand. This difficulty could have been avoided by taking necessary precautions earlier in the calculation, but the present procedure is considerably more concise and proper care is taken in handling the divergence.

We wish to carry out the momentum integration prior to the frequency sum for the correlation function. When the frequency sum is carried out first, the resulting momentum integration is very narrowly peaked about the Fermi surface and the integrand decreases exponentially for large $|p-p_F|$. Exact calculations for the normal metal, where the mathematics is quite tractable, verify these comments.[21] For the superconductor, one may subtract the normal metal result from $Q(\omega_0)$ so that the order of summation and integration may be interchanged. The magnitude of the vectors $\mathbf{p}$ in the integrand are approximated by p_F and ξ by $v_F(p-p_F)$. Here, we evaluate the integral

$$\int d\xi\,\mathcal{G}(q,k,\omega_l')\tau^{(3)}\mathcal{G}(q,k,\omega_l) = \pi(\tilde{\epsilon}_l+\tilde{\epsilon}_l')^{-1}[\tau^{(3)}\tau^{(3)}\tau^{(3)} + \tilde{\epsilon}_l'^{-1}(i\tilde{\omega}_l'+\tilde{\Delta}_l'\tau^{(1)})\tau^{(3)}(i\tilde{\omega}_l+\tilde{\Delta}_l\tau^{(1)})\tilde{\epsilon}_l^{-1}]. \quad (51)$$

The function $\tilde{\epsilon}_l$ is equal to $(\tilde{\omega}_l^2+\tilde{\Delta}_l^2)^{1/2}$. With Eq. (51) one now has

$$Q(\omega_0) = \frac{p_F^3}{4m}T\sum_l\int\frac{d\Omega}{(2\pi)^2}(\omega_l+\omega_l')^2\mathbf{pp}\frac{1}{\tilde{\epsilon}_l+\tilde{\epsilon}_l'} \times\left(1-\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right) \quad (52)$$

and in the zero-frequency limit, $\omega_0=0$, the correlation function vanishes identically. The thermal gradient has an influence only on the excited single-particle excitations and there is no thermal counterpart of the Meissner effect.

The thermal conductivity is proportional to the frequency derivative of the retarded correlation function in the zero-frequency limit[22]

$$\kappa = \lim_{\omega_1\to 0} i\frac{\partial Q^R(\omega_1)}{\partial\omega_1}. \quad (53)$$

The function $Q^R(\omega_1)$ is the analytic continuation of $Q(\omega_0)$ that is free of singularities in the upper-half plane of ω_1. $Q(\omega_0)$ is recovered for $\omega_1=i\omega_0$. $Q^R(t-t')$ is the causal propagator of the current operator $\mathbf{h}$ and must vanish for $t<t'$. It follows that its Fourier transform is analytic in the upper-half ω_1 plane.

The mathematical technique for evaluating a retarded correlation function from one defined in the temperature representation for the discrete frequencies ω_0 has been discussed in detail by Abrikosov, Gor'kov, and Dzyaloshinskii[23] and by Maki.[24] The frequency sum of ω_l is converted to an integral in the complex ω plane by multiplying the summand with $\tan\omega/2T$ and surrounding the poles of this function by a contour. The contour is then deformed to embrace the branch cuts of $\tilde{\epsilon}$ and $\tilde{\epsilon}'$. Exploiting the periodicity of $\tan\omega/2T$ and the discreteness of ω_0, one may show that the contour integral surrounding the branch cuts of $\tilde{\epsilon}'$ may be discarded provided that those over $\tilde{\epsilon}$ are multiplied by 2. This is done even though the original frequency sum is formally divergent. The resulting function is analytic in the upper-half plane of the continuous variable ω_1. As a result, we have in the low ω_1 limit

$$Q^R(\omega_1) = \frac{i3N}{8mT}\int\frac{d\Omega}{2\pi}\hat{p}\hat{p}\int dE \times\left[\tanh\left(\frac{E+\omega_1}{2T}\right)-\tanh\left(\frac{E}{2T}\right)\right] \times\left(1+\frac{|\tilde{E}|^2-|\tilde{\Delta}|^2}{|\tilde{E}^2-\tilde{\Delta}^2|}\right)\frac{E^2}{\mathrm{Im}(\tilde{E}^2-\tilde{\Delta}^2)^{1/2}} \quad (54)$$

in this approximation, so that

$$\kappa = \frac{3N}{16mT^2}\int\frac{d\Omega}{2\pi}\hat{p}\hat{p}\int_0^\infty\frac{dE\,\mathrm{sech}^2(E/2T)E^2}{\mathrm{Im}(\tilde{E}^2-\tilde{\Delta}^2)^{1/2}} \times\left(1+\frac{|\tilde{E}|^2-|\tilde{\Delta}|^2}{|\tilde{E}^2-\tilde{\Delta}^2|}\right). \quad (55)$$

[21] A. A. Abrikosov, L. P. Gor'kov, and I. Y. Dzyaloshinskii, in *Quantum Field Theoretical Methods* (Pergamon Press, Inc., London, 1965), p. 312.

[22] V. Ambegaokar, in *Brandeis Lectures* (W. A. Benjamin, Inc., New York, 1963), Vol. 2, p. 423.

[23] See Ref. 21, p. 311.

[24] K. Maki, Progr. Theoret. Phys. (Kyoto) **31**, 378 (1964).

The functions $\tilde{E}(\theta,E)$ and $\tilde{\Delta}(\theta,E)$ are the analytic continuation of $\tilde{\omega}_l$ and $\tilde{\Delta}_l$ to the physical region of real energies. The element of solid angle is $d\Omega=\sin\theta\, d\theta\, d\phi$.

In the low-density limit, $n_v\to 0$, the expression in the parentheses of Eq. (55) reduces to $2\theta(E-\Delta)$, if we neglect the effect of bound states. With Eq. (38) of Sec. II, we finally have

$$\kappa=\frac{3N}{4mT^2}\int\frac{d\Omega}{2\pi}\hat{p}\hat{p}\int_\Delta^\infty\frac{dE\,E^2\,\mathrm{sech}^2(E/2T)}{1/\tau_0+1/\tau_n(\theta,E)} \tag{56}$$

in the limit $n_v\to 0$, provided scattering-in terms are neglected. The addition of $1/\tau_0$ to $1/\tau_n(\theta,E)$ takes into account a dilute background of spinless spherical impurities with relaxation time τ_0. For a normal metal, the integral of Eq. (56) is elementary, and we have

$$\kappa_n=N\pi^2\tau_0T/3m\,, \tag{57}$$

which is the correct result, if τ_0 is the transport lifetime of the electrons. It is assumed here that the impurities give rise to s-wave scattering only.

The scattering-in terms are very important for electron scattering by a vortex and they may not be neglected. The momentum variables appearing in the functions of Eq. (49) are approximated by their value on the Fermi surface except for $\xi\simeq v_F(p-p_F)$. ξ only appears in the Green's functions. Of the two momenta q and k, only the polar angle $\theta=\tan^{-1}q/k$ remains as a variable.

The case of heat conduction parallel to the external field is easiest to deal with. The scattering self-energy has only two components proportional to $\tau^{(0)}$ and $\tau^{(1)}$, when the coupling angle of the T matrix is equal to $\frac{1}{4}\pi$. For scattering of single-particle excitations by a vortex, the coupling angle χ_μ, is in fact $\frac{1}{4}\pi$.[9] The function M is then the sum of two parts

$$M=[M_3(\theta,\omega_l,\omega_l')\tau^{(3)}+M_2(\theta,\omega_l,\omega_l')\tau^{(2)}]p_F\sin\theta\,. \tag{58}$$

After performing the integration over ξ, Eq. (49) becomes

$$\begin{aligned}M_3\tau^{(3)}&+M_2\tau^{(2)}\\&=\tau^{(3)}(\omega_l+\omega_l')+n_vm(2\pi)^{-2}\int_0^{2\pi}d\phi\,\pi(\tilde{\epsilon}_l+\tilde{\epsilon}_l')^{-1}\\&\quad\times\mathcal{T}(\phi,\theta,\omega_l')[\tau^{(3)}(M_3\tau^{(3)}+M_2\tau^{(2)})\tau^{(3)}\\&\quad+(i\tilde{\omega}_l'+\tilde{\Delta}_l'\tau^{(1)})(M_3\tau^{(3)}+M_2\tau^{(2)})(i\tilde{\omega}_l+\tilde{\Delta}_l\tau^{(1)})\\&\quad\times\tilde{\epsilon}_l^{-1}\tilde{\epsilon}_l'^{-1}]\mathcal{T}(-\phi,\,\theta,\,\omega_l)\,,\end{aligned} \tag{59}$$

with $\phi=\cos^{-1}\bar{q}\cdot\bar{q}'$.

Solution of the equation for M is straightforward and we write the results below:

$$M_3D_l=(\omega_l+\omega_l')\left[\tilde{\epsilon}_l+\tilde{\epsilon}_l'+\frac{1}{2\tau_a}\left(1+\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right)+\frac{1}{2\tau_b}\frac{\tilde{\omega}_l\tilde{\Delta}_l'-\tilde{\omega}_l'\tilde{\Delta}_l}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right], \tag{60}$$

$$M_2D_l=(\omega_l+\omega_l')\left[\frac{1}{2\tau_a}\frac{\tilde{\omega}_l\tilde{\Delta}_l'-\tilde{\omega}_l'\tilde{\Delta}_l}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}+\frac{1}{2\tau_b}\left(1-\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right)\right], \tag{61}$$

and

$$D_l=\tilde{\epsilon}_l+\tilde{\epsilon}_l'+\frac{1}{\tau_a}\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}+\frac{1}{\tau_b}\frac{\tilde{\omega}_l\tilde{\Delta}_l'-\tilde{\omega}_l'\tilde{\Delta}_l}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\,. \tag{62}$$

The lifetimes τ_a and τ_b are defined as follows:

$$\begin{aligned}\frac{1}{2\tau_a(\theta,\omega_l,\omega_l')}&=\frac{n_vm}{4\pi}\int_0^{2\pi}d\phi\,[\mathcal{T}^{(0)}(\phi,\theta,\omega_l')\mathcal{T}^{(0)}(-\phi,\,\theta,\,\omega_l)\\&\quad-\mathcal{T}^{(1)}(\phi,\theta,\omega_l')\mathcal{T}^{(1)}(-\phi,\,\theta,\,\omega_l)]\,,\end{aligned} \tag{63}$$

and

$$\begin{aligned}\frac{1}{2\tau_b(\theta,\omega_l,\omega_l')}&=\frac{n_vm}{4\pi}i\int_0^{2\pi}d\phi\,[\mathcal{T}^{(0)}(\phi,\theta,\omega_l')\mathcal{T}^{(1)}(-\phi,\,\theta,\,\omega_l)\\&\quad-\mathcal{T}^{(1)}(\phi,\theta,\omega_l')\mathcal{T}^{(0)}(-\phi,\,\theta,\,\omega_l)]\,.\end{aligned} \tag{64}$$

Substituting the expressions of Eqs. (60) and (62) into Eq. (47) for the vector component $K_z(q,k,\omega_l,\omega_l')$ and performing the necessary integration to obtain the correlation function, one has

$$Q(\omega_0)_{zz}=\frac{p_F^3}{4m}T\sum_l\int\frac{d\Omega}{(2\pi)^2}\hat{p}_z\hat{p}_z(\omega_l+\omega_l')^2\times\left(1-\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right)D_l^{-1}. \tag{65}$$

Again it is not the temperature correlation function but the retarded function that is desired. Equation (65) is almost identical to Eq. (52) for the correlation function without scattering-in terms. Only the denominators, $\tilde{\epsilon}_l+\tilde{\epsilon}_l'$ and D_l, differ. One may show that Eq. (54) still holds for the thermal conductivity if the denominator $\mathrm{Im}(\tilde{E}^2+\tilde{\Delta}^2)^{1/2}$ is replaced by

$$\mathrm{Im}(\tilde{E}^2-\tilde{\Delta}^2)^{1/2}-\frac{(|\tilde{E}^2|-|\tilde{\Delta}|^2)/|\tilde{E}^2-\tilde{\Delta}^2|}{2\tau_a[\theta,\,-i(E-i\epsilon),\,-i(E+i\epsilon)]}+\frac{2\,\mathrm{Im}\tilde{E}^*\tilde{\Delta}/|\tilde{E}^2-\tilde{\Delta}^2|}{2\tau_b[\theta,\,-i(E-i\epsilon),\,-i(E+i\epsilon)]}\,.$$

In the limit $n_v\to 0$, the contribution from the term proportional to τ_b^{-1} is of order n_v^2 and may be neglected.

The lifetime $\tau_a[\theta, -i(E-i\epsilon), -i(R+i\epsilon)]$, which for simplicity we denote as $\tau_a(\theta,E)$, is given by

$$\frac{1}{\tau_a(\theta,E)}=\frac{n_v m}{2\pi}\int_0^{2\pi} d\phi\,[|T^{(0)}(\phi,\theta,E+i\epsilon)|^2 - |T^{(1)}(\phi,\theta,E+i\epsilon)|^2], \quad (66)$$

which is identically $1/\tau_e(\theta,E)$ of Ref. 8. Therefore, Eq. (56) for κ still holds in the limit $B\to 0$ if $1/\tau_n(\theta,E)$ is replaced by $1/\tau_n(\theta,E)-1/\tau_e(\theta,E)=2/\tau_s(\theta,E)$.

It is interesting that the vortex scattering contribution to κ_{zz} does not vanish, so that this component of the conductivity tensor is finite even in the absence of atomic impurities. In the quasiparticle picture, a vortex couples the particlelike and holelike excitations of the same energy. Since the scattering is very narrowly peaked, one can summarize an event by a diagram of Fig. 3. An incident particlelike excitation $(q^2+k^2>p_F^2)$ impinges on a vortex from the bottom left-hand corner of the diagram. It may scatter elastically, carrying energy and momentum to the top right-hand corner, or be diverted to the alternate channel. In the latter case, the excitation is holelike, propagating in the incident direction, but carrying energy back towards the source. κ_{zz} thus remains finite even though the superconductor is translationally invariant in the z direction. This effect is purely a superconducting one and could not be observed in a normal metal.

The diagonal components of κ in the (x,y) plane are approximated very well by taking $\mathbf{N}$ parallel to $\mathbf{q}$. The asymmetry in electron-vortex scattering is quite small. The only modification introduced is to change $d\phi$ to $\cos\phi\, d\phi$ in the defining equations for the collision frequencies τ_a^{-1} and τ_b^{-1}. Since the integrand is narrowly peaked in the forward direction, one may replace $\cos\phi$ by 1 to lowest order in $[p_F\xi(T)]^{-1}$. Equation (56) for κ still holds if $1/\tau_n(\theta,E)$ is replaced by $2/\tau_s(\theta,E)$.

Since $1/\tau_n(\theta,E)$ diverges as a result of the BA effect, it is very gratifying to see $2/\tau_s(\theta,E)$ appear in its place. This latter collision frequency is independent of the BA effect and is finite except for an integrable singularity at $E=\Delta$. In the limit $B\to 0$, the ratio of the diagonal components of κ in the superconducting state to their normal-metal values are

$$\frac{\kappa}{\kappa_n}=\frac{9}{4\pi^2T^3}\int_\Delta^\infty \frac{d\Omega}{2\pi}\hat{p}\hat{p}\frac{dE\,E^2\,\mathrm{sech}^2(E/2T)}{1+2\tau_0/\tau_s(\theta,E)}. \quad (67)$$

The thermal conductivity has nonzero off-diagonal components κ_{xy} and κ_{yx} due to the asymmetry of electron-vortex scattering. The vector $\mathbf{N}$ has two orthogonal components

$$\mathbf{N}=N'\mathbf{q}+N''\hat{z}\times\mathbf{q}, \quad (68)$$

which together satisfy Eq. (49). The two coupled equations for N' and N'' are

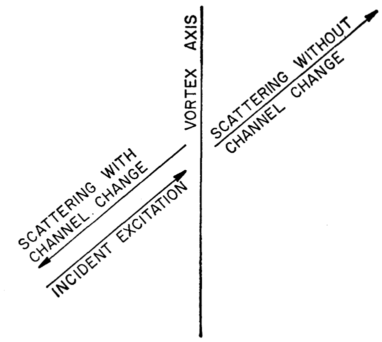

FIG. 3. Schematic diagram of scattering of particlelike excitations by a vortex.

$$N'(\theta,\omega_l,\omega_l') =1+n_v(2\pi)^{-2}\int d\mathbf{q}'\,\mathcal{T}(\phi,\theta,\omega_l')\mathcal{G}(q',k,\omega_l') \times[N'(\theta,\omega_l,\omega_l')\cos\phi-N''(\theta,\omega_l,\omega_l')\sin\phi] \times\mathcal{G}(q',k,\omega_l)\mathcal{T}(-\phi,\theta,\omega_l) \quad (69)$$

and

$$N''(\theta,\omega_l,\omega_l') =n_v(2\pi)^{-2}\int d\mathbf{q}'\,\mathcal{T}(\phi,\theta,\omega_l')\mathcal{G}(q',k,\omega_l') \times[N'(\theta,\omega_l,\omega_l')\sin\phi+N''(\theta,\omega_l,\omega_l')\cos\phi] \times\mathcal{G}(q',k,\omega_l)\mathcal{T}(-\phi,\theta,\omega_l). \quad (70)$$

The component N'' is much smaller than N' and its appearance in Eq. (69) is ignored, while Eq. (70) is kept intact. It follows that N' is known, having been previously determined. Calculation of N'' is straightforward but the algebra is cumbersome. We simply quote the result for the off-diagonal components of κ in the limit $B\to 0$:

$$\kappa_{xy}=\frac{3N}{16mT^2}\int\frac{d\Omega}{2\pi}\hat{p}_x\hat{p}_x \times\int_\Delta^\infty\frac{dE\,E^2\,\mathrm{sech}^2(E/2T)1/\tau_e'(\theta,E)}{[1/\tau_0+2/\tau_s(\theta,E)]^2}, \quad (71)$$

where $1/\tau_e'(\theta,E)$ is given by Eq. (66) with $d\phi$ replaced by $\sin\phi\, d\phi$. When $1/\tau_0\to 0$, the ratio of the off-diagonal components of κ to the diagonal components is of the order $[p_F\xi(T)]^{-1}$. For pure niobium and vanadium, this quantity is less than 10^{-2}.

V. ULTRASONIC ATTENUATION

Ultrasonic propagation is described by the ion-displacement vector $\boldsymbol{\phi}(x,t)$ which satisfies the wave equation

$$\frac{MN}{Z}\frac{\partial^2\phi}{\partial t^2}-\frac{Np_F{}^3}{5m}[3\nabla(\nabla\cdot\phi)-\nabla\times(\nabla\times\phi)]+\langle\mathbf{f}\rangle=0, \tag{72}$$

where M is the ion's mass and Z is its valence.[25,26] The last term in Eq. (72) is the thermodynamic average of the residual force produced by the electrons on the ion in a frame moving with the lattice. The force operator is

$$\mathbf{f}(\mathbf{x},t)=m(d/dt)\mathbf{J}(\mathbf{x},t)+\nabla\cdot\tau(\mathbf{x},t), \tag{73}$$

where J is the electric current operator and τ is the electronic stress tensor, given by

$$\tau(\mathbf{x},t)=-\sum_\alpha (4m)^{-1}(\nabla-\nabla')(\nabla-\nabla')\times\psi_\alpha{}^\dagger(\mathbf{x}',t)\psi_\alpha(\mathbf{x},t)\,|_{\mathbf{x}=\mathbf{x}'}. \tag{74}$$

The force operator f may be calculated in the equilibrium lattice if the interaction Hamiltonian

$$H_I=-\int d\mathbf{x}\;\phi(\mathbf{x},t)\cdot\mathbf{f}_I(\mathbf{x},t) \tag{75}$$

is introduced.

The vector field $\phi(\mathbf{x},t)$ is conveniently decomposed into a longitudinal and a transverse part

$$\phi=\phi_l+\phi_t, \tag{76}$$

where

$$\nabla\times\phi_l=0\quad\text{and}\quad\nabla\cdot\phi_t=0. \tag{77}$$

The longitudinal and transverse parts of ϕ are expressed in terms of potentials as follows:

$$\phi_l(\mathbf{x},t)=-\nabla\xi(\mathbf{x},t) \tag{78}$$

and

$$\phi_t(\mathbf{x},t)=\nabla\times\mathbf{V}(\mathbf{x},t). \tag{79}$$

Equation (72) can now be decomposed into a longitudinal and transverse part in terms of ξ and $\mathbf{V}$. The interaction Hamiltonian can also be expressed in terms of the potentials

$$H_I=-\int d\mathbf{x}\,[\xi(\mathbf{x},t)\nabla\cdot\mathbf{f}_I(\mathbf{x},t)+\mathbf{V}(\mathbf{x},t)\cdot\nabla\times\mathbf{f}_I(\mathbf{x},t)]. \tag{80}$$

The inhomogeneous term in the equations for ξ and $\mathbf{V}$ is evaluated to lowest nonvanishing order in perturbation theory. The zeroth-order term vanishes and the next term is linear in the potentials. Equations (78) and (79) are evaluated for plane sound waves

$$\xi(x)=\xi_\alpha\exp[i(p_1x_\alpha-\omega_1t)] \tag{81}$$

and

$$\mathbf{V}(\mathbf{x})=V_{\alpha\beta}\boldsymbol{\varepsilon}_\beta\exp[i(p_1x_\alpha-\omega_1t)]. \tag{82}$$

Here α indicates the direction of propagation of the wave and β indicates the direction of polarization of the transverse wave. The unit polarization vector $\boldsymbol{\varepsilon}_\beta$ points at right angles to the direction of propagation, $\boldsymbol{\varepsilon}_\beta\cdot\mathbf{p}_1=0$. With the modifications made, the equations become

$$(p_1{}^2v_L{}^2-\omega_1{}^2)\xi_\alpha=-(\rho_{\text{ion}}p_1{}^2)^{-1}\times\langle[\nabla\cdot\mathbf{f}_I,\xi_\beta\nabla\cdot\mathbf{f}_I+V_{\beta\gamma}\boldsymbol{\varepsilon}_\gamma\cdot\nabla\times\mathbf{f}_I]\rangle(p_1,\omega_1) \tag{83}$$

and

$$(p_1{}^2v_T{}^2-\omega_1{}^2)V_{\alpha\beta}=-(\rho_{\text{ion}}p_1{}^2)^{-1}\times\langle[\epsilon_\beta\cdot\nabla\times\mathbf{f}_I,V_{\gamma e}\boldsymbol{\varepsilon}_e\cdot\nabla\times\mathbf{f}_I+\xi_\gamma\nabla\cdot\mathbf{f}_I]\rangle(p_1,\omega_1), \tag{84}$$

where v_L and v_T are the velocity of sound for longitudinal and transverse waves, $\rho_{\text{ion}}=MN/Z$ is the ionic mass density, and

$$v_L{}^2=3Zp_F{}^2/5Mm=3v_T{}^2. \tag{85}$$

The terms on the right-hand side of Eqs. (83) and (84) are the Fourier transforms of the retarded propagator for spatial derivatives of $\mathbf{f}$, i.e.,

$$\langle[A,B]\rangle(\omega)=-i\int_{-\infty}^{t}dt'e^{i\omega(t-t')}\langle[A(t),B(t')]\rangle. \tag{86}$$

In a superconductor containing spinless atomic impurities, only those correlation functions which couple to themselves are nonvanishing. For example, the correlation function $\langle[\nabla\cdot f_I,\nabla\times f_I]\rangle$ which couples longitudinal and transverse modes must vanish for a system that is time reversal invariant. Some of these functions are no longer zero for a superconductor containing vortices. The asymmetry in the scattering of electrons by vortices couples some ultrasonic modes. This coupling is very small and will not be calculated here, although we mention its occurrence.

The attenuation constant for longitudinal waves is equal to twice the imaginary part of the wave vector, so that

$$\alpha_L=-\text{Re}[\langle[\nabla\cdot\mathbf{f}_I,\nabla\cdot\mathbf{f}_I]\rangle(p_1,\omega_1)/i\rho_{\text{ion}}\omega_1v_Lp_1{}^2] \tag{87}$$

and there are two of these quantities, one each for sound waves propagating parallel and perpendicular to the vortices. For the transverse waves, the attenuation is

$$\alpha_T=-\text{Re}[\langle[\nabla\times\mathbf{f}_I\cdot\boldsymbol{\varepsilon},\,\nabla\times\mathbf{f}_I\cdot\boldsymbol{\varepsilon}]\rangle(p_1,\omega_1)/i\rho_{\text{ion}}\omega_1v_Tp_1{}^2] \tag{88}$$

of which there are also two, for waves polarized along and at right angles to the external field.

Calculations are performed in the temperature representation and we begin with the time-ordered correlation function for longitudinal waves, which is defined as

$$R_L(1,2)=\langle T_\tau\nabla\cdot\mathbf{f}_I(1)\nabla\cdot\mathbf{f}_I(2)\rangle. \tag{89}$$

Employing the local conservation law for electrons, we may equate the divergence of $\mathbf{f}$ to $m\partial^2n/\partial\tau^2+\nabla\cdot\nabla\cdot\tau$, where n is the electron-density operator.

The function R is evaluated in terms of single-particle Green's functions and it is Fourier transformed in space and time. The position of the vortices is averaged over

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[26] L. P. Kadanoff and I. I. Falko, Phys. Rev. **136**, A1170 (1964).

so that the medium becomes translationally invariant. In the limit $p_1 \to 0$, the correlation function becomes

$$R_L(\mathbf{p}_1,\omega_0) = -T\sum_l \int d\mathbf{p}\,(2\pi)^{-3}[m\omega_0^2 + m^{-1}(\mathbf{p}_1\cdot\mathbf{p})^2] \times \mathrm{Tr}[\tau^{(3)}L_L(\mathbf{q},k,\omega_l,\omega_l')] \quad (90)$$

with

$$L_L(q,k,\omega_l,\omega_l') \equiv \mathcal{G}(q,k,\omega_l')U_L(q,k,\omega_l,\omega_l')\mathcal{G}(q,k,\omega_l), \quad (91)$$

where the function $U_L(q,k,\omega_l,\omega_l')$ satisfies

$$U_L(q,k,\omega_l,\omega_l') = [m\omega_0^2 + m^{-1}(\mathbf{p}_1\cdot\mathbf{p})^2]\tau^{(3)} + n_v(2\pi)^{-2}\int d\mathbf{q}'\,\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l')\mathcal{G}(q',k,\omega_l') \times U_L(q',k,\omega_l,\omega_l')\mathcal{G}(q',k,\omega_l)\mathcal{T}(\mathbf{q}',\mathbf{q};k,\omega_l). \quad (92)$$

As they now stand, Eqs. (90) and (92) are incorrect since the long-range part of the Coulomb repulsion between electrons has been ignored. Kadanoff and Falko wave calculated the necessary modifications to the theory due to this longitudinal fluctuation, and we conform to their results by replacing $m\omega_0^2$ where it appears in the square brackets of Eqs. (90) and (92) with $-p_1^2p_F^2/3m$.[26] The ratio of these two quantities is of the order Zm/M, the ratio of the ion to the electron plasma frequencies.

In the transverse case $\nabla\cdot\mathbf{f}_I$ is replaced by $\nabla\times f_I$ in the defining equation for $R_T(1,2)$, where

$$\nabla\times\mathbf{f}_I \simeq \nabla\times(\nabla\cdot\tau),$$

since the transverse electric current is screened by the superconducting electrons and is unimportant at low frequencies and long wavelengths. Omitting details, we summarize the basic equations for the correlation function

$$R_T(p,\omega_0) = -T\sum_l \int d\mathbf{p}\,(2\pi)^{-3}m^{-1}(\mathbf{p}_1\cdot\mathbf{p})(\mathbf{p}_1\times\mathbf{p}\cdot\boldsymbol{\varepsilon}) \times \mathrm{Tr}[\tau^{(3)}L_T(q,k,\omega_l,\omega_l')], \quad (93)$$

where

$$L_T(q,k,\omega_l,\omega_l') \equiv \mathcal{G}(q',k,\omega_l')U_T(q,k,\omega_l,\omega_l')\mathcal{G}(q,k,\omega_l) \quad (94)$$

and

$$U_T(q,k,\omega_l,\omega_l') = m^{-1}(\mathbf{p}_1\cdot\mathbf{p})(\mathbf{p}_1\times\mathbf{p}\cdot\boldsymbol{\varepsilon})\tau^{(3)} + n_V(2\pi)^{-2}\int d\mathbf{q}'\,\mathcal{G}(q',k,\omega_l')\mathcal{T}(\mathbf{q},\mathbf{q}';k,\omega_l') \times U_T(\mathbf{q}',k,\omega_l,\omega_l')\mathcal{T}(\mathbf{q}',\mathbf{q};k,\omega_l)\mathcal{G}(q',k,\omega_l). \quad (95)$$

The simplest mode to consider is the longitudinal one which propagates along the magnetic field direction. The bare vertex for this mode is independent of ϕ, and is equal to $p_1^2p_F^2m^{-1}(-\frac{1}{3}+\cos^2\theta)\tau^{(3)}$, where we have set $p=p_F$ for the reasons given in Sec. IV. The polar coordinates ϕ,θ are defined in the usual way for a set of axes with z lying along the direction of the external field.

The manipulations for obtaining $R_L^{\parallel}(p,\omega_0)$ are identical to those of Sec. IV, and we quote the result

$$R_L^{\parallel}(p,\omega_0) = -\frac{p_F^5p_1^4}{m}T\sum_l \int \frac{d\Omega}{(2\pi)^2}(\cos^2\theta-\tfrac{1}{3})^2 \times\left(1-\frac{\tilde{\omega}_l\tilde{\omega}_l'+\tilde{\Delta}_l\tilde{\Delta}_l'}{\tilde{\epsilon}_l\tilde{\epsilon}_l'}\right)D_l^{-1}. \quad (96)$$

The bare vertices for propagation of longitudinal waves perpendicular to the field are

$$p_1^2p_F^2m^{-1}(-\tfrac{1}{3}+\sin^2\theta\cos^2\phi)\tau^{(3)}$$

and

$$p_1^2p_F^2m^{-1}(-\tfrac{1}{3}+\sin^2\theta\sin^2\phi)\tau^{(3)}.$$

Instead of solving for $U_L^x(q,k,\omega_l,\omega_l')$ and $U_L^y(q,k,\omega_l,\omega_l')$ for waves traveling in the x and y direction, one may solve for the linear combination

$$U_L^+(q,k,\omega_l,\omega_l') = \tfrac{1}{2}(U_L^x+U_L^y) \quad (97)$$

and

$$U_L^-(q,k,\omega_l,\omega_l') = \tfrac{1}{2}(U_L^x-U_L^y). \quad (98)$$

The bare vertex corresponding to the plus process is independent of ϕ and an expression for U_L^+ may be written immediately. In the minus case, the bare vertex is $\frac{1}{2}p_1^2p_F^2\sin^2\theta\cos2\phi$, and writing

$$U_L^-(q,k,\omega_l,\omega_l') = u_L'(\theta,\omega_l,\omega_l')\cos2\phi,$$

we have the equation

$$u_L^-(\theta,\omega_l,\omega_l')\cos2\phi = \tfrac{1}{2}p_1^2p_F^2m^{-1}\sin^2\theta\cos2\phi + n_v\int d\mathbf{q}'\,(2\pi)^{-2}\mathcal{G}(q',k,\omega_l')\mathcal{T}(\phi',\theta,\omega_l')u_L^-(\theta,\omega_l,\omega_l') \times[\cos2\phi\cos2\phi'-\sin2\phi\sin2\phi'] \times\mathcal{T}(-\phi',\theta,\omega_l)\mathcal{G}(q',k,\omega_l). \quad (99)$$

In the square brackets of Eq. (99), the term $\sin2\phi'\times\sin2\phi$ may be neglected as the scattering is almost symmetrical about $\phi'=0$. The function $\cos2\phi'$ is approximated by 1 since the scattering is forward peaked. It follows that $R_L^{\perp}(p_1,\omega_0)$ is obtained from the expression for $R_L^{\parallel}(p_1,\omega_0)$ by replacing $(\cos^2\theta-\frac{1}{3})^2$ with

$$(\sin^2\theta\cos^2\phi-\tfrac{1}{3})^2.$$

In a similar fashion, one may calculate $R_T^{\parallel}$ and $R_T^{\perp}$ for transverse modes polarized parallel and perpendicular to **B**.

The procedure for obtaining the attenuation constant α from R is very similar to that employed for obtaining the thermal conductivity κ from Q.

The results of this section are summarized below for $\mathbf{B}\rightarrow 0$, for $T\simeq T_c$ and neglecting bound states.

$$\frac{\alpha_j^i}{\alpha_n}=\frac{45}{16T}\int_0^\pi d\theta\,\sin\theta\,F_j^i(\theta)\int_\Delta^\infty\frac{dE\,\mathrm{sech}^2(E/2T)}{1+\tau_0/\tau_s(\theta,E)}, \quad (100)$$

where $F_L^{\parallel}(\theta)=(\cos^2\theta-\frac{1}{3})^2$ refers to longitudinal waves propagating parallel to the vortices, $F_L^{\perp}(\theta)=\frac{3}{8}\sin^4\theta-\frac{1}{3}\sin^2\theta+\frac{1}{9}$ refers to longitudinal waves propagating at right angles to the vortices, and $F_T^{\parallel}(\theta)=\frac{1}{6}\sin^4\theta$, and $F_T^{\perp}(\theta)=\frac{2}{3}\sin^2\theta\cos^2\theta$ refer to transverse waves polarized parallel and perpendicular to $\mathbf{B}$.

VI. COMPARISON WITH EXPERIMENT

With the results of Secs. IV and V, comparison with experiment is possible in the limit $B\rightarrow 0$, and $T\rightarrow T_c$. The contribution of the bound states to the thermal conductivity and ultrasonic attenuation has been neglected in Eqs. (67) and (100) and to ensure validity of the theory, temperature must be restricted to a range where the scattering states are heavily populated.

As B increases from zero, theory predicts a decrease in the thermal conductivity and ultrasonic attenuation. An anomaly of this kind was observed by Forgan and Gough in ultrasonic attenuation measurements performed on clean niobium.[7] A similar decrease in the ultrasonic attenuation was witnessed by Tsuda *et al.*, who also performed measurements on niobium.[27] Recently, Sinclair and Liebowitz have carried out some very accurate measurements on the dip in vanadium, so that evidence exists for the effect in both primary superconductors.[10] Although the interpretation of thermal conductivity experiments is complicated by phonon contributions to the heat flux and by a phonon contribution to electron lifetime,[28] Lowell and Sousa have observed what appears to be a dip in the thermal conductivity in experiments performed on clean niobium.[29]

When the thermal and directional averages over E and θ have been performed, the cross section in Eq. (100) is replaced by a temperature-dependent parameter $a_j^i(T)$. This result was first presented by Forgan and Gough in an analysis of their experimental data. The indices i and j are similar to those appearing in Eq. (100), where i runs over $\parallel$ and $\perp$ and j over L and T.

We have calculated the four $a_j^i(T)$ by numerical quadrature employing the results of Ref. 9. Our calculations predict an average vortex cross section of 210 Å near T_c for longitudinal sound waves in vanadium, propagating along the direction of $\mathbf{B}$. The measurements of Sinclair and Liebowitz suggest a cross section of 240 Å, which is in good agreement with the predictions of our simple model.

The cross sections for the other sound modes have also been calculated and we find that for longitudinal waves propagating at right angles to $\mathbf{B}$, $a_L^{\perp}=360$ Å, whereas the cross sections for transverse waves polarized along and at right angles to B are $a_T^{\parallel}=410$ Å and $a_T^{\perp}=245$ Å. It would be interesting to see if these predictions were verified experimentally.

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